Module 2

Division Algorithm

Given integers a and b with $b>0$ there exist unique integers q and r satisfying.

$$a = qb + r, \quad 0 \le r < b$$

The integers q and r are called the quotient and ~~reminder~~ remainder in the division of a and b.

If $a = 33$ and $b = 4$, then

$$33 = 8 \times 4 + 1$$

$$\Rightarrow q = 8 \text{ and } r = 1$$

If $b=2$ the possible ~~remaind~~ remainders are $r=0$ and $r=1$.

When $r=0$, the integer a has the form $a = 2q$ and is called even

when $r=1$, the integer a has the form $a = 2q+1$. and is called odd.

(1). Show that the square of an integer leaves the remainder 0 or 1 upon division by 4.

Solution

Every integer a is of the form either $a = 2q$ or $a = 2q+1$.

If $a = 2q$

$a^2 = (2q)^2 = 4q^2$, leaves the remainder 0 when divided by 4.

If $a = 2q+1$

$a^2 = (2q+1)^2$

$= 4q^2 + 4q + 1$

$= 4(q^2+q)+1$

$= 4k+1$, leaves the remainder 1 when divided by 4.

Q7. Show that the square of any odd integer is of the form $8k+1$

Solution

By division algorithm, any ~~integer~~ ~~odd~~ integer is representable as one of the four forms $4q$, $4q+1$, $4q+2$, $4q+3$. Here $4q+1$ and $4q+3$ are odd.

$(4q+1)^2 = 16q^2 + 8q + 1 = 8(2q^2+q)+1 = 8k+1.$

$(4q+3)^2 = 16q^2 + 24q + 9 = 8(2q^2+3q+1)+1 = 8k+1$

(3) ~~A~~ Prove that the expression $\frac{a(a^2+2)}{3}$ is an integer for all $a \geq 1$.

Solution

According to division algorithm, every a is of the form $3q$, $3q+1$ or $3q+2$

If $a = 3q$, $\frac{a(a^2+2)}{3} = \frac{3q((3q)^2+2)}{3}$

$= q(9q^2+2)$, is an integer

If $a = 3q+1$, $\frac{a(a^2+2)}{3} = \frac{(3q+1)((3q+1)^2+2)}{3}$

$= (3q+1)(3q^2+2q+1)$, an integer

If $a = 3q+2$,

$\frac{a(a^2+2)}{3} = \frac{(3q+2)((3q+2)^2+2)}{3}$

$= (3q+2)(3q^2+4q+2)$, an integer.

(4) Show that cube of any integer has one of the forms, $9k$, $9k+1$, or $9k+8$.

Solution.

By division algorithm, every integer a is of the form $3q$, $3q+1$, or $3q+2$.

If $a = 3q$

$$\cancel{a^3 = 9q}. \quad a^3 = (3q)^3$$

$$= 27q^3$$

$$= 9(3q^3)$$

$$= 9k$$

If $a = 3q+1$

$$a^3 = (3q+1)^3$$

$$= 27q^3 + 27q^2 + 9q + 1$$

$$= 9(3q^3 + 3q^2 + q) + 1$$

$$= 9k+1$$

If $a = 3q+2$

$$a^3 = (3q+2)^3$$

$$= 27q^3 + 54q^2 + 36q + 8$$

$$= 9(3q^3 + 2q^2 + 4q) + 8$$

$$= 9k+8$$

Greatest Common Divisor (GCD)

Definition

An integer b is said to be divisible by an integer $a \neq 0$, in symbols a/b, if there exists some integer c such that $b = ac$.

We write $a \nmid b$ to indicate that b is not divisible by a.

Definition

If a and b are arbitrary integers, then an integer d is said to be a common divisor of a and b if both d/a and d/b.

Definition

Let a and b be given integers, with at least one of them different from zero. The greatest common divisor of a and b, denoted by $\gcd(a,b)$ is the positive integer d satisfying the following.

ⓐ $d|a$ and $d|b$

ⓑ If $c|a$ and $c|b$ then $c \le d$.

The positive divisors of -12 are $1, 2, 3, 4, 6, 12$ whereas those of 30 are $1, 2, 3, 5, 6, 10, 15, 30$. The positive common divisors of -12 and 30 are $1, 2, 3, 6$, Because 6 is the largest of these integers, $\gcd(-12, 30) = 6$.

$\gcd(-5, 5) = 5$

$\gcd(8, 17) = 1$

$\gcd(-8, -36) = 4$.

$\gcd(-12, 30) = 6 = (-12)2 + 30 \cdot 1$

$\gcd(-8, -36) = 4 = (-8)4 + (-36)(-1)$

Definition

Given integers a and b, not both of which are zero, there exist integers x and y such that $\gcd(a,b) = ax + by$.

Definition

Two integers a and b not both of which are zero, are said to be relatively prime whenever $\gcd(a,b) = 1$

Euclidean Algorithm

Let a and b be two integers. By divisional Algorithm

$$a = q_1 b + r_1, \quad 0 \le r_1 < b.$$

If it happens that $r_1 = 0$, then b/a and $\gcd(a,b) = b$.

When $r_1 \neq 0$, divide b by r_1 to produce integers q_2 and r_2 satisfying.

$$b = q_2 r_1 + r_2, \quad 0 \le r_2 < r_1$$

If $r_2 = 0$, then we stop. otherwise.

$$r_1 = q_3 r_2 + r_3, \quad 0 \le r_3 < r_2$$

The division process continuous until some zero remainder appears, say at the $(n+1)^{th}$ stage where r_{n-1} is divided by r_n.

The result is the following system of equations.

$$a = q_1 b + r_1 \quad , \quad 0 < r_1 < b$$

$$b = q_2 r_1 + r_2 \quad , \quad 0 < r_2 < r_1$$

$$r_1 = q_3 r_2 + r_3 \quad , \quad 0 < r_3 < r_2$$

$$\vdots$$

$$r_{n-2} = q_n r_{n-1} + r_n \, , \quad 0 < r_n < r_{n-1}$$

$$r_{n-1} = q_n r_n + 0$$

The last non zero remainder r_n is the $\gcd(a,b)$.

(1) Find $\gcd(12378, 3054)$

Solution

$$12378 = 4 \cdot 3054 + 162$$

$$3054 = 18 \cdot 162 + 138$$

$$162 = 1 \cdot 138 + 24$$

$$138 = 5 \cdot 24 + 18$$

$$24 = 1 \cdot 18 + 6$$

$$18 = 3 \cdot 6 + 0$$

$$\gcd(12378, 3054) = 6.$$

Now, $6 = 24 - 18$

$$= 24 - (138 - 5 \cdot 24)$$

$$= 6 \cdot 24 - 138$$

$= 6(162-138) - 138$

$= 6 \cdot 162 - 7 \cdot 138$

$= 6 \cdot 162 - 7\,(3054 - 18 \cdot 162)$

$= 132 \cdot 162 - 7 \cdot 3054$

$= 132\,(12378 - 4 \cdot 3054) - 7 \cdot 3054$

$= 132 \cdot 12378 + (-535)\, 3054$

$6 = \gcd(12378, 3054) = 12378x + 3054y$, where

$x = 132$ and $y = -535$.

(2) Find $\gcd(143, 227)$

Solution

$227 = 1 \cdot 143 + 84$

$143 = 1 \cdot 84 + 59$

$84 = 1 \cdot 59 + 25$

$59 = 2 \cdot 25 + 9$

$\cancel{25 = 2 \cdot 18 + 7}$ $\quad 25 = 2 \cdot 9 + 7$

$\cancel{18 = 2 \cdot 7 + 4}$ $\quad 9 = 1 \cdot 7 + 2$

$\cancel{7 = 1 \cdot 4 + 3}$ $\quad 7 = 3 \cdot 2 + 1$

$\cancel{4 = 1 \cdot 3 + 1}$ $\quad 2 = 1 \cdot 2 + 0$

$\cancel{3 = 1 \cdot 3 + 0}$

$\gcd(143, 227) = 1$

Now, $1 = 7 - 3\cdot 2$

$= 7 - 3\cdot(9 - 1\cdot 7)$

$= 2\cdot 7 - 3\cdot 9$

$= 2\cdot(25 - 2\cdot 9) - 3\cdot 9$

$= 2\cdot 25 - 7\cdot 9$

$= 2\cdot 25 - 7\cdot(59 - 2\cdot 25)$

$= 16\cdot 25 - 7\cdot 59$

$= 16\cdot(84 - 1\cdot 59) - 7\cdot 59$

$= 16\cdot 84 - 23\cdot 59$

$= 16\cdot 84 - 23\cdot(143 - 1\cdot 84)$

$= 39\cdot 84 - 23\cdot 143$

$= 39\cdot(227 - 1\cdot 143) - 23\cdot 143$

$= 39\cdot 227 - 62\cdot 143$

$1 = 39 \cdot 227 + (-62) \cdot 143$

$1 = \gcd(227, 143) = 227x + 143y$ where

$x = 39$ and $y = -62$.

(3) Find $\gcd(306, 657)$

Solution

$657 = 2 \cdot 306 + 45$

$306 = 6 \cdot 45 + 36$

$45 = 1 \cdot 36 + 9$

$36 = 4 \cdot 9 + 0$

$\gcd(306, 657) = 9$

Now. $9 = 45 - 1 \cdot 36$

$= 45 - 1 \cdot (306 - 6 \cdot 45)$

$= 7 \cdot 45 - 1 \cdot 306$

$= 7 \cdot (657 - 2 \cdot 306) - 1 \cdot 306$

$= 7 \cdot 657 - 15 \cdot 306$

$9 = \gcd(657, 306) = 657x + 306y$ where.

$x = 7$ and $y = -15$.

(4) Use the Euclidean Algorithm to obtain integers x and y satisfying the following.

$\gcd(119, 272) = 119x + 272y$.

Solution

$$272 = 2 \cdot 119 + 34$$

$$119 = 3 \cdot 34 + 17$$

$$34 = 2 \cdot 17 + 0$$

Now, $17 = 119 - 3 \cdot 34$

$$= 119 - 3 \cdot (272 - 2 \cdot 119)$$

$$= 7 \cdot 119 - 3 \cdot 272.$$

$$17 = \gcd(272, 119) = 7 \cdot 119 + (-3) 272$$

$\gcd(272, 119) = 119x + 272y$ where

$x = 7$ and $y = -3$.

Exercises

(1). Find $\gcd(272, 1479)$

(2) Use the Euclidean Algorithm to obtain integers x and y satisfying the following.

(a) $\gcd(56, 72) = 56x + 72y$

(b) $\gcd(24, 138) = 24x + 138y$

(c) $\gcd(1769, 2378) = 1769x + 2378y.$

Diophantine Equation

The linear Diophantine equation $ax+by=c$ has a solution if and only if $d \mid c$, where $d = \gcd(a,b)$. If x_0, y_0 is any particular solution of this equation, then all other solutions are given by $x = x_0 + \left(\frac{b}{d}\right)t$, $y = y_0 - \left(\frac{a}{d}\right)t$.

where t is an arbitrary integer.

(i) Solve the Diophantine equation $172x + 20y = 1000$.

Solution

Using Euclidean algorithm on 172 and 20

$$172 = 8 \cdot 20 + 12.$$

$$20 = 1 \cdot 12 + 8$$

$$12 = 1 \cdot 8 + 4$$

$$8 = 2 \cdot 4$$

$\gcd(172, 20) = 4.$

Since $4 \mid 1000$, a solution to this equation exists.

$$4 = 12 - 8$$

$$= 12 - (20 - 1 \cdot 12)$$

$$= 2 \cdot 12 - 20$$

$$= 2 \cdot (172 - 8 \cdot 20) - 20$$

$$= 2 \cdot 172 - \cancel{16 \cdot 20}\ 17 \cdot 20$$

$$4 = 2 \cdot 172 + (-17) \cdot 20$$

Multiply this relation with 250.

$$1000 = 4 \cdot 250 = 250\,[2 \cdot 172 + (-17)\,20]$$

$$= 500 \cdot 172 + (-4250)\,20.$$

$x_0 = 500$ and $y_0 = -4250.$

The other solutions are.

$$x = x_0 + \left(\frac{b}{d}\right)t = 500 + \left(\frac{20}{4}\right)t = 500 + 5t$$

$$y = y_0 - \left(\frac{a}{d}\right)t = -4250 - \left(\frac{172}{4}\right)t = -4250 - 43t.$$

2. Solve the linear Diophantine equation

$24x + 138y = 18.$

Solution

Using Euclidean algorithm on 24 and 138.

$$138 = 5 \cdot 24 + 18$$

$$24 = 1 \cdot 18 + 6$$

$$18 = 3 \cdot 6.$$

$\gcd(24, 138) = 6.$

Since $6 / 18$, a solution to this equation exist.

$$6 = 24 - 18$$

$$6 = 24 - (138 - 5 \cdot 24)$$

$$6 = 6 \cdot 24 - 138$$

$$6 = 6 \cdot 24 + (-1)\,138$$

Multiply the relation with 3,

$$18 = 3 \cdot 6 = 3\left[6 \cdot 24 + (-1)\, 138\right]$$

$$18 = 18 \cdot 24 + (-3) \cdot 138$$

$$18 = 24x + 138y \text{, where}$$

$$x_0 = 18 \text{ and } y_0 = -3$$

The other solutions are ;

$$x = x_0 + \left(\frac{b}{d}\right)t = 18 + \left(\frac{138}{6}\right)t$$

$$x = 18 + 23t$$

$$y = y_0 - \left(\frac{a}{d}\right)t = -3 - \left(\frac{24}{6}\right)t$$

$$y = -3 - 4t.$$

3. Solve the linear Diophantine equation

$$54x + 21y = 906$$

Solution

$$\cancel{54 = 4 \cdot 21 +}$$

$$54 = 2 \cdot 21 + 12$$

$$21 = 1 \cdot 12 + 9$$

$$12 = 1 \cdot 9 + 3$$

$$9 = 3 \cdot 3$$

$$\gcd(54, 21) = 3.$$

Since $3 | 906$, a solution to this equation exist.

Now, $3 = 12 - 1 \cdot 9$

$= 12 - (21 - 12)$

$= 2 \cdot 12 - 21$

$= 2 \cdot (54 - 2 \cdot 21) - 21$

$= 2 \cdot 54 - 5 \cdot 21$

Multiply the relation with 302

$906 = 3 \cdot 302 = 302 [2 \cdot 54 - 5 \cdot 21]$

$906 = 604 \cdot 54 - 1510 \cdot 21$

$906 = 54x + 21y$ where

$x_0 = 604$ and $y_0 = -1510$

The other solutions are

$x = x_0 + \left(\frac{b}{d}\right)t = 604 + \left(\frac{21}{3}\right)t$

$x = 604 + 7t$

$y = y_0 - \left(\frac{a}{d}\right)t = -1510 - \left(\frac{54}{3}\right)t$

$y = -1510 - 18t$

4. check whether a solution for the Diophantine equation $6x + 51y = 22$ exist or not.

Solution

$$51 = 8 \cdot 6 + 3$$

$$6 = 2 \cdot 3$$

$\gcd(51, 6) = 3$

But 3 does not divides 22, so the solution for Diophantine equation does not exist.

Exercises

1. Solve the Diophantine equations.

(1) $56x + 72y = 40$

(2) $221x + 35y = 11$

(3) $18x + 5y = 48$

(4) $123x + 360y = 99$

(5) $158x - 57y = 7$.

Congruence

Let n be a fixed positive integer. Two integers a and b are said to be congruent modulo n, symbolized by $a \equiv b \pmod{n}$.

if n divides the difference $a-b$, that is $a-b = kn$ for some integer k.

If $n \nmid a-b$ then a is incongruent to b modulo n, and write $a \not\equiv b \pmod{n}$.

For example $24 \equiv 3 \pmod 7$

Since $24-3 = 21 = 3 \cdot 7$

$-31 \equiv 11 \pmod 7$

Since $-31-11 = -42 = -6 \cdot 7$

But $25 \not\equiv 12 \pmod 7$ since

$25-12 = 13 =$ not a multiple of 7.

Properties of Congruences

ⓐ $a \equiv a \pmod{n}$

ⓑ If $a \equiv b \pmod{n}$ then $b \equiv a \pmod{n}$

ⓒ If $a \equiv b \pmod{n}$ and $b \equiv c \pmod{n}$ then $a \equiv c \pmod{n}$

ⓓ If $a \equiv b \pmod{n}$ and $c \equiv d \pmod{n}$, then $a+c \equiv b+d \pmod{n}$ and $ac \equiv bd \pmod{n}$

(e) If $a \equiv b \pmod{n}$ then $a+c \equiv b+c \pmod{n}$ and $ac \equiv bc \pmod{n}$.

(f) If $a \equiv b \pmod{n}$ then $a^k \equiv b^k \pmod{n}$ for any positive integer k.

(1) Show that 41 divides $2^{20}-1$.

<u>Solution</u>

$$2^5 = 32 \equiv -9 \pmod{41}$$

$$2^5 \equiv -9 \pmod{41}$$

$$(2^5)^4 \equiv (-9)^4 \pmod{41}$$

$$2^{20} \equiv (-9)^2(-9)^2 \pmod{41}$$

$$2^{20} \equiv (81)(81) \pmod{41}$$

But $81 \equiv -1 \pmod{41}$

$$81 \cdot 81 \equiv 1 \pmod{41}$$

$$\therefore\ 2^{20} \equiv 1 \pmod{41}$$

$$2^{20}-1 \equiv 0 \pmod{41}$$

$\therefore$ 41 divides $2^{20}-1$.

(2) ~~Show~~ Find the remainders when 2^{50} and 41^{65} are divided by 7.

<u>Solution</u>

$$2^2 \equiv 1 \pmod{7}$$

$$(2^2)^{25} \equiv 1^{25} \pmod{7}$$

$$2^{50} \equiv 1 \pmod{7}$$

The remainder when 2^{50} divided by 7 is 1.

Also $\quad 41 \equiv -1 \pmod 7$

$$(41)^{65} \equiv (-1)^{65} \pmod 7$$

$$41^{65} \equiv -1 \pmod 7$$

The remainder when 41^{65} divided by 7 is -1.

(3) Find the remainder when dividing the sum

$1! + 2! + 3! + 4! + \cdots + 99! + 100!$ by 12.

Solution

We have ~~4! ≡ 0 (mod~~

$$4! = 24 \equiv 0 \pmod{12}$$

$$5! = 5 \cdot 4! = 5 \cdot 0 \equiv 0 \pmod{12}$$

$$6! \equiv 6 \cdot 5 \cdot 4! \equiv 0 \pmod{12}$$

$$\vdots$$

$$100! \equiv 0 \pmod{12}$$

$$1! + 2! + 3! + 4! + \cdots + 99! + 100!$$

$$\equiv 1! + 2! + 3! + 0 + \cdots + 0 \pmod{12}$$

$$\equiv 1 + 2 + 6 \pmod{12}$$

$$\equiv 9 \pmod{12}$$

$\therefore$ The remainder is 9.

Linear Congruence

The linear Congruence $ax \equiv b \pmod{n}$ has a solution if and only if d/b where $d = \gcd(a, n)$. If d/b then it has d mutually incongruent solutions modulo n.

The congruence is equivalent to the linear Diaphantine equation $ax = ny$

$ax - b = ny$

$ax - ny = b$

This equation can be solved if d/b.

The other solutions are.

$$x = x_0 + \frac{n}{d}t$$

$$y = y_0 + \frac{a}{d}t$$

(i) Solve the linear congruence $18x \equiv 30 \pmod{42}$

Solution

$18x \equiv 30 \pmod{42}$

$18x - 30 = 42y$

$18x - 42y = 30$

Also, $42 = 2 \cdot 18 + 6$

$18 = 3 \cdot 6.$

so, $\gcd(18, 42) = 6$

Since 6 divides 30, there exists a solution.

By inspection one solution is $x = 4$.

The six solutions are.

$$x \equiv 4 + \left(\frac{42}{6}\right)t$$

$$x \equiv 4 + 7t, \quad t = 0, 1, 2, 3, 4, 5$$

i.e $x \equiv 4, 11, 18, 25, 32, 39 \pmod{42}$

(2) Solve the linear Congruence $9x \equiv 21 \pmod{30}$.

Solution

$$9x \equiv 21 \pmod{30}$$

$$9x - 21 = 30y.$$

$$9x - 30y = 21.$$

Also $30 = 3 \cdot 9 + 3$

$$9 = 3 \cdot 3.$$

$\gcd(9,30) = 3.$

Sine 3 divides 21 there exists a solution.

Here $3 = 30 - 3 \cdot 9.$

$21 = 7 \cdot 3 = 7[30 - 3 \cdot 9]$

$21 = 7 \cdot 30 - 21 \cdot 9.$

We have $21 = 9x - 30y.$

$21 = 9(-21) - 30(-7)$

$\therefore x_0 = -21, \quad y_0 = -7.$

The three solutions are.

$x = -21 + \left(\frac{30}{3}\right)t$

$x = -21 + 10t. \quad , t = 0, 1, 2$

$x = -21, -11, -1 \pmod{30}.$

(3) Solve the linear Congruence $25x \equiv 15 \pmod{29}$

Solution

$25x \equiv 15 \pmod{29}$

$25x - 15 = 29y$

$25x - 29y = 15.$

Also $29 = 1 \cdot 25 + 4$

$25 = 6 \cdot 4 + 1$

$4 = 1 \cdot 4.$

$\gcd(25, 29) = 1$

Here $1 = 25 - 6 \cdot 4$

$= 25 - 6\,[29 - 1 \cdot 25]$

$1 = 7 \cdot 25 - 6 \cdot 29$

We have $25x - 29y = 15$, multiply by 15

$15 = 15\,[7 \cdot 25 - 6 \cdot 29]$

$15 = 105 \cdot 25 - 90 \cdot 29$

$\therefore\ x = 105$ and $y = 90$

The solutions are

$x = 105 + \left(\frac{29}{1}\right)t \quad , t = 0$

$x = 105 + 29t \quad , t = 0$

$x = 105$

Exercises

Solve the following linear congruences.

(1) $5x \equiv 2 \pmod{26}$

(2) $6x \equiv 15 \pmod{21}$

(3) $36x \equiv 8 \pmod{102}$

(4) $34x \equiv 60 \pmod{98}$

(5) $140x \equiv 133 \pmod{301}$.

Recurrence Relation

The general form of a first order linear homogeneous recurrence relation with constant coefficients is

$a_{n+1} = d\,a_n,\ n \geq 0$, where d is a constant. Since a_{n+1} depends only on its immediate predecessor, the relation is said to be first order.

The unique solution of the recurrence relation $a_{n+1} = d\,a_n,\ n \geq 0$, d is a constant with initial condition $a_0 = A$ is

$$\boxed{\begin{array}{l} a_n = A d^n,\ n \geq 0. \\ a_n = a_0 d^n \end{array}}$$

(i) Solve the recurrence relation $a_n = 7a_{n-1}$, where $n \geq 1$ and $a_2 = 98$

Solution

The solution is

$$a_n = a_0 7^n \quad (\text{Here } d = 7)$$

given $a_2 = 98$

$$a_0 7^2 = 98 \Rightarrow a_0 = \frac{98}{49} = 2.$$

$$\therefore\ a_n = 2 \cdot 7^n,\ n \geq 0.$$

(2) Solve $a_{n+1} - 1.5 a_n = 0, \ n \geq 0$

Solution

$$a_{n+1} = 1.5 a_n$$
$$d = 1.5$$

Solution $a_n = a_0 (1.5)^n$

(3) Solve $3a_{n+1} - 4a_n = 0, \ n \geq 0, \ a_1 = 5$

Solution

$$3a_{n+1} = 4a_n$$
$$a_{n+1} = \frac{4}{3} a_n$$
$$d = \frac{4}{3}.$$

solution $a_n = a_0 (4/3)^n.$

Given $a_1 = 5$

$$a_0 (4/3) = 5$$
$$a_0 = \frac{15}{4}$$
$$\therefore \quad a_n = \left(\frac{15}{4}\right)(4/3)^n$$

(4) Find the unique solution of the recurrence relation $6a_n - 7a_{n-1} = 0, \ n \geq 1, \ a_3 = 343$

Solution

$$6a_n = 7a_{n-1}$$
$$a_n = \left(\frac{7}{6}\right) a_{n-1}$$
$$d = 7/6.$$

Solution is $a_n = a_0\left(\frac{7}{6}\right)^n$

Given $a_3 = 343$

$$a_0\left(\frac{7}{6}\right)^3 = 343$$

$$a_0 = \frac{343 \times 216}{343} = 216$$

$$\therefore a_n = 216\left(\frac{7}{6}\right)^n$$

(5) Find a_{12} if $a_{n+1}^2 = 5a_n^2$, $a_n > 0$ for $n \geq 0$ and $a_0 = 2$

Solution

The recurrence relation is not linear in a_n. Let $b_n = a_n^2$, then the relation becomes

$$b_{n+1} = 5b_n, \quad b_0 = a_0^2 = 4$$

The solution for b_n is $b_n = b_0 5^n$

$$b_n = 4 \cdot 5^n.$$

$$\therefore a_n = 2(\sqrt{5})^n,$$

$$a_{12} = 2(\sqrt{5})^{12}$$

$$a_{12} = 31,250$$

(6) Solve the relation $a_n = n \cdot a_{n-1}$, where $n \geq 1$ and $a_0 = 1$.

Solution

This relation is a recurrence relation with variable coefficient.

$a_0 = 1$

$a_1 = 1 . a_0 = 1$

$a_2 = 2 . a_1 = 2 . 1 = 2!$

$a_3 = 3 . a_2 = 3 . 2 . 1 = 3!$

$a_4 = \cancel{4} . a_3 = 4 . 3 . 2 . 1 = 4!$

$\therefore \quad a_n = n!$

Exercises

① Solve the recurrence relations

(a) $4a_n - 5a_{n-1} = 0, \quad n \geq 1$

(b) $2a_n - 3a_{n-1} = 0, \quad n \geq 1, \quad a_4 = 81$

Second Order Linear Homogeneous Recurrence Relation with Constant Coefficients.

The general form of second order homogeneous recurrence relation with constant coefficients is

$$C_0 a_n + C_1 a_{n-1} + C_2 a_{n-2} = 0, \quad n \geq 2$$

Take $a_n = c r^n$

$$C_0 c r^n + C_1 c r^{n-1} + C_2 c r^{n-2} = 0.$$

$$(C_0 r^2 + C_1 r + C_2) c r^{n-2} = 0.$$

$$C_0 r^2 + C_1 r + C_2 = 0.$$

The equation $C_0 r^2 + C_1 r + C_2 = 0$ is called the characteristic equation. Let r_1 and r_2 be the roots of the equation.

Case (A) Distinct real roots.

If r_1 and r_2 are the distinct real roots, the solution is

$$\cancel{a_n = c_1(r_1^n) + c_2(r_2^n)}$$

$$a_n = c_1(r_1^n) + c_2(r_2^n)$$

(1) Solve the recurrence relation

$a_n + a_{n-1} - 6a_{n-2} = 0, \quad n \geq 2, \quad a_0 = -1, \; a_1 = 8.$

Solution

If $a_n = cr^n$

$$\cancel{a_{n+1} = cr^{n+1}}$$

$$a_{n-1} = cr^{n-1}$$

$$a_{n-2} = cr^{n-2}.$$

Equation becomes $cr^n + cr^{n-1} - 6cr^{n-2} = 0$

$$cr^{n-2}(r^2 + r - 6) = 0$$

$$r^2 + r - 6 = 0$$

$$(r+3)(r-2) = 0$$

$$r = -3, 2.$$

$$\therefore \; a_n = c_1(2^n) + c_2(-3)^n$$

If $a_0 = -1$,

$$a_0 = c_1(2^0) + c_2(-3)^0 = c_1 + c_2 = -1$$

If $a_1 = 8$

$$a_1 = c_1(2^1) + c_2(-3)^1 = 2c_1 - 3c_2 = 8$$

$$c_1 + c_2 = -1 \quad \text{——(1)}$$

$$2c_1 - 3c_2 = 8 \quad \text{——(2)}$$

$$(1) \times 2 \Rightarrow 2c_1 + 2c_2 = -2 \quad \text{——(3)}$$

$$(2) \Rightarrow 2c_1 - 3c_2 = 8 \quad \text{——(4)}$$

$$(3) \mp (4) \Rightarrow 5c_2 = -10$$

$$c_2 = -2$$

$$(1) \Rightarrow c_1 - 2 = -1$$

$$c_1 = 1$$

$$\therefore \; a_n = 2^n - 2(-3)^n.$$

(2) Solve the recurrence relation

$$2a_n = 7a_{n-1} - 3a_{n-2} \; ; \; a_0 = 2, \; a_1 = 5$$

<u>Solution</u>

If $a_n = c\gamma^n$

$$a_{n-1} = c\gamma^{n-1}$$

$$a_{n-2} = c\gamma^{n-2}$$

Equation becomes $2c\gamma^n = 7c\gamma^{n-1} - 3c\gamma^{n-2}$

$$c\gamma^{n-2}\left[2\gamma^2 - 7\gamma + 3\right] = 0$$

$$2\gamma^2 - 7\gamma + 3 = 0$$

$$r = \frac{7 \pm \sqrt{49-24}}{4}$$

$$r = \frac{7 \pm 5}{4}$$

$$r = 3, \frac{1}{2}$$

$$\therefore \quad a_n = c_1 (3^n) + c_2 \left(\frac{1}{2}\right)^n$$

If $a_0 = 2$, $\quad a_0 = c_1 (3^0) + c_2 (1/2)^0 = c_1 + c_2 = 2.$

If $a_1 = 5$, $\quad a_1 = c_1 (3^1) + c_2 (1/2)^1 = 3c_1 + \frac{1}{2}c_2 = 5$

$$c_1 + c_2 = 2 \quad \text{——} (1)$$

$$3c_1 + \frac{1}{2}c_2 = 5 \quad \text{——} (2)$$

$(2) \times 2 \quad 6c_1 + c_2 = 10 \quad \text{——} (3)$

$(3) - (1) \Rightarrow \quad 5c_1 = 8$

$$c_1 = \frac{8}{5}$$

$(1) \Rightarrow \quad c_2 = 2 - \frac{8}{5} = \frac{2}{5}$

$$a_n = \frac{8}{5}(3^n) + \frac{2}{5}(1/2)^n$$

(3) Solve the recurrence relation $F_{n+2} = F_{n+1} + F_n$.

where $n \geq 0$, and $F_0 = 0$, $F_1 = 1$

Solution

If $F_n = c r^n$

$$F_{n+1} = c r^{n+1}$$

$$F_{n+2} = c r^{n+2}.$$

Equation becomes

$$c r^{n+2} = c r^{n+1} + c r^{n}$$

$$c r^{n}\left[r^{2} - r - 1\right] = 0$$

$$r^{2} - r - 1 = 0$$

$$r = \frac{1 \pm \sqrt{1+4}}{2}$$

$$r = \frac{1 \pm \sqrt{5}}{2}$$

$$F_n = c_1\left(\frac{1+\sqrt{5}}{2}\right)^{n} + c_2\left(\frac{1-\sqrt{5}}{2}\right)^{n}$$

$$F_0 = c_1 + c_2 = 0 \quad \text{———} (1)$$

$$F_1 = c_1\left(\frac{1+\sqrt{5}}{2}\right) + c_2\left(\frac{1-\sqrt{5}}{2}\right) = 1 \quad \text{———} (2)$$

$$(1) \times (1+\sqrt{5}) \Rightarrow c_1(1+\sqrt{5}) + c_2(1+\sqrt{5}) = 0 \quad \text{———} (3)$$

$$(2) \times 2 \Rightarrow c_1(1+\sqrt{5}) + c_2(1-\sqrt{5}) = 2 \quad \text{———} (4)$$

$$(3) - (4) \Rightarrow c_2(2\sqrt{5}) = -2.$$

$$c_2 = \frac{-1}{\sqrt{5}}$$

$$(1) \Rightarrow c_1 = \frac{1}{\sqrt{5}}$$

$$F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n}$$

Repeated Real roots

(1) Solve the recurrence relation

$a_{n+2} = 4a_{n+1} - 4a_n, \quad n \geq 0, \quad a_0 = 1, a_1 = 3$

Solution

If $a_n = cr^n$

$a_{n+1} = cr^{n+1}$

$a_{n+2} = cr^{n+2}$

$$cr^{n+2} = 4cr^{n+1} - 4cr^n$$

$$cr^n\left[r^2 - 4r + 4\right] = 0$$

$$r^2 - 4r + 4 = 0$$

$$(r-2)^2 = 0$$

$$r = 2, 2.$$

$$\therefore a_n = c_1(2^n) + c_2 n(2^n)$$

$$a_0 = c_1 + \cancel{c_2} = 1 \quad \text{———} (1)$$

$$a_1 = c_1(2) + c_2(2) = 3$$

$$2c_2 = 1$$

$$c_2 = 1/2.$$

$$a_n = (2^n) + \frac{1}{2}n(2^n)$$

(2) Solve the recurrence relation.

$a_n - 6a_{n-1} + 9a_{n-2} = 0, \quad n \geq 2, \quad a_0 = 5, a_1 = 12.$

Solution

$$\cancel{a_n = cr^n,\ a_{n+1} = cr^{n+1},\ a_{n+2} = cr^{n+2}}$$

$$a_n = cr^n,\ a_{n-1} = cr^{n-1},\ a_{n-2} = cr^{n-2}$$

$$cr^n - 6cr^{n-1} + 9cr^{n-2} = 0$$

$$cr^{n-2}\left[r^2 - 6r + 9\right] = 0$$

$$r^2 - 6r + 9 = 0$$

$$(r-3)^2 = 0$$

$$r = 3, 3$$

$$a_n = c_1(3^n) + c_2 n(3^n)$$

$$a_0 = c_1 = 5 \quad \text{--- (1)}$$

$$a_1 = c_1(3) + c_2(3) = 12 \quad \text{--- (2)}$$

$$15 + 3c_2 = 12$$

$$3c_2 = -3$$

$$c_2 = -1$$

$$a_n = 5(3^n) - n(3^n)$$

Complex Roots

(i) Solve the recurrence relation

$$a_n = 2(a_{n-1} - a_{n-2}),\quad n \geq 2,\ a_0 = 1,\ a_1 = 2$$

Solution

$$a_n = cr^n,\ a_{n-1} = cr^{n-1},\ a_{n-2} = cr^{n-2}$$

$$cr^n = 2(cr^{n-1} - cr^{n-2})$$

$$cr^{n-2}(r^2 - 2r + 2) = 0$$

$$r = \frac{2 \pm \sqrt{4-8}}{2}$$

$$r = 1 \pm i$$

$$a_n = c_1 (1+i)^n + c_2 (1-i)^n$$

$$a_0 = c_1 (1+i)^0 + c_2 (1-i)^0 = c_1 + c_2 = 1 \quad \text{---} (1)$$

$$a_1 = c_1 (1+i) + c_2 (1-i) = 2 \quad \text{---} (2)$$

$$(1) \times (1+i) \Rightarrow \quad c_1 (1+i) + c_2 (1+i) = 1+i \quad \text{---} (3)$$

$$(2) - (3) \Rightarrow \quad c_2 (-2i) = (1-i)$$

$$c_2 = \frac{1-i}{-2i} = \frac{1}{2}(i+1)$$

$$c_1 = 1 - c_2$$

$$= 1 - \frac{(i+1)}{2}$$

$$= \frac{1}{2}(1-i)$$

$$a_n = \frac{1}{2}(1-i)(1+i)^n + \frac{1}{2}(1+i)(1-i)^n.$$

Exercises

Solve the recurrence relations

(1) $a_n = 5a_{n-1} + 6a_{n-2}, \; n \geq 2, \; a_0 = 1, \; a_1 = 3$

(2) $2a_{n+2} - 11a_{n+1} + 5a_n = 0, \; n \geq 0, \; a_0 = 2, \; a_1 = -8$

(3) $a_{n+3} + a_n = 0, \; n \geq 0, \; a_0 = 0, \; a_1 = 3$

(4) $a_n + 2a_{n-1} + 2a_{n-2} = 0, \; n \geq 2, \; a_0 = 1, \; a_1 = 3.$

Non homogeneous recurrence relation

Consider the non homogeneous first order relation $a_1 + C_1 a_{n-1} = k r^n$, where k is a constant and $n \in Z^+$. If r^n is not a solution of the associated homogeneous relation $a_n + C_1 a_{n-1} = 0$, then $a_n^{(P)} = A r^n$ where A is a constant. If r^n is a solution of the corresponding homogeneous relation, then $a_n^{(P)} = B n r^n$, B is a constant.

Consider the non homogeneous second order relation $a_n + C_1 a_{n-1} + C_2 a_{n-2} = k r^n$, where k is a constant.

(a) If r^n is not a solution of the homogeneous relation $a_n^{(P)} = A r^n$

(b) If $a_n^{(h)} = C_1 r^n + C_2 r_1^n$ where $r_1 \neq r$,

$a_n^{(P)} = B n r^n$ where B is a constant

(c) If $a_n^{(h)} = (C_1 + C_2 n) r^n$

$a_n^{(P)} = C n^2 r^n$, C is a constant.

The solution is $a_n = a_n^{(h)} + a_n^{(P)}$

① Solve the recurrence relation

$$a_n - 3a_{n-1} = 5(7^n), \quad n \geq 1, \text{ and } a_0 = 2$$

Solution

$$a_n = 3a_{n-1}$$

$$a_n^{(h)} = C_0(3^n)$$

Since $f(n) = 5(7^n)$, take $a_n^{(P)} = A(7^n)$

$\cancel{a_n^{(P+1)} = A}$

$$a_{n-1}^{(P)} = A(7^{n-1})$$

Equation becomes $A(7^n) - 3(A(7^{n-1})) = 5(7^n)$

$$A\,7^{n-1}(7-3) = 5\,7^{n-1}.7$$

$$4A = 35$$

$$A = \frac{35}{4}$$

$$a_n^{(P)} = \frac{35}{4}(7^n)$$

$$\therefore a_n = a_n^{(h)} + a_n^{(P)}$$

$$a_n = C_0(3^n) + \frac{35}{4}(7^n)$$

Given $a_0 = 2$.

$$a_0 = C_0(3^0) + \frac{35}{4}(7^0)$$

$$a_0 = C_0 + \frac{35}{4} = 2$$

$$C_0 = 2 - \frac{35}{4} = \frac{-27}{4}$$

$$a_n = \frac{-27}{4}(3^n) + \frac{35}{4}(7^n)$$

② Solve the recurrence relation

$$a_n - 3a_{n-1} = 5(3^n), \quad n \geq 1, \quad a_0 = 2$$

Solution

$$a_n - 3a_{n-1} = 0$$

$$a_n = 3a_{n-1}$$

$$a_n^{(h)} = C_0(3^n)$$

Since $f(n) = 5(3^n)$. take $a_n^{(p)} = Bn(3^n)$

$$a_{n-1}^{(p)} = B(n-1)(3^{n-1})$$

Equation becomes $Bn(3^n) - 3B(n-1)(3^{n-1}) = 5(3^n)$

$$3^{n-1}\left(Bn3 - 3B(n-1)\right) = 5\,3^{n-1}\,3$$

$$3B(n-n+1) = 15$$

$$B = 5$$

$$a_n^{(p)} = 5n(3^n)$$

$$a_n = a_n^{(h)} + a_n^{(p)}$$

$$a_n = C_0(3^n) + 5n(3^n).$$

Given $a_0 = 2$

$$a_0 = C_0(3^0) + 0 = 2$$

$$C_0 = 2$$

$$a_n = 2(3^n) + 5n(3^n)$$

$$a_n = (2+5n)(3^n)$$

(3) ~~(5)~~ Solve the recurrence relation

$a_{n+2} + 3a_{n+1} + 2a_n = 3^n, \ n \geq 0, \ a_0 = 0, \ a_1 = 1$

Solution

Let $a_n = cr^n$ be the solution of the homogeneous equation $a_{n+2} + 3a_{n+1} + 2a_n = 0$.

$$cr^{n+2} + 3cr^{n+1} + 2cr^n = 0$$

$$r^n(cr^2 + 3cr + 2c) = 0$$

$$r^2 + 3r + 2 = 0$$

$$(r+1)(r+2) = 0$$

$$r = -1, -2$$

$$\therefore \ a_n^{(h)} = A(-1)^n + B(-2)^n$$

Since $f(n) = 3^n$, take $a_n^{(p)} = D(3^n)$

$$D(3^{n+2}) + 3D(3^{n+1}) + 2D(3^n) = 3^n$$

$$3^n[9D + 9D + 2D] = 3^n$$

$$20D = 1$$

$$D = \frac{1}{20}$$

$$\therefore \ a_n^{(p)} = \frac{1}{20}(3^n)$$

$$\therefore \ a_n = a_n^{(h)} + a_n^{(p)}$$

$$a_n = A(-1)^n + B(-2)^n + \frac{1}{20}(3^n).$$

(4) Solve the recurrence relation

$$a_{n+2} - 8a_{n+1} + 16a_n = 8(5^n) + 6(4^n), \; n \geq 0$$

$a_0 = 12$, $a_1 = 5$

Solution

Let $a_n = c r^n$ be the solution of the homogeneous equation $a_{n+2} - 8a_{n+1} + 16a_n = 0$, Then $a_{n+1} = c r^{n+1}$ and $a_{n+2} = c r^{n+2}$

$$\therefore \; c r^{n+2} - 8 c r^{n+1} + 16 c r^n = 0$$

$$c r^n [r^2 - 8r + 16] = 0$$

$$r^2 - 8r + 16 = 0$$

$$(r-4)^2 = 0$$

$$r = 4, 4$$

$$\therefore \; a_n^{(h)} = A(4^n) + Bn(4^n)$$

Here $f(n) = 8(5^n) + 6(4^n)$, take

$$a_n^{(P)} = c(5^n) + Dn^2(4^n)$$

$$a_{n+1}^{(P)} = c(5^{n+1}) + D(n+1)^2(4^{n+1})$$

$$a_{n+2}^{(P)} = c(5^{n+2}) + D(n+2)^2(4^{n+2})$$

Substituting in the equation.

$$c(5^{n+2}) + D(n+2)^2(4^{n+2}) - 8[c(5^{n+1}) + D(n+1)^2(4^{n+1})]$$

$$+ 16[c(5^n) + Dn^2(4^n)] = 8(5^n) + 6(4^n)$$

Comparing Coefficients of (5^n) and (4^n)

$(5^n)[\cancel{c\,25 + D(n+2)^2}$

$$(5^n)\left[25C\,\cancel{*} - 40C + 16C\right] = 8(5^n)$$

$$25C - 40C + 16C = 8$$

$$C = 8$$

$$(4^n)\left[D(n+2)^2 16 - 8(n+1)^2 D(4) + 16Dn^2\right] = 6\,(4^n)$$

$$D\left[16n^2 + 64n + 64 - 32n^2 - 64n - 32 + 16n^2\right] = 6$$

$$D(32) = 6$$

$$D = \frac{6}{32} = \frac{3}{16}$$

$$a_n = a_n^{(h)} + a_n^{(P)}$$

$$a_n = A\,4^n + Bn\,(4^n) + 8\,(5^n) + \frac{3}{16}\,n^2\,(4^n)$$

Given $a_0 = 12$

$$a_0 = A + 8 = 12$$

$$A = 4$$

Given $a_1 = 5$

$$a_1 = 4A + 4B + 40 + \frac{3}{4} = 5$$

$$4B + 56 + \frac{3}{4} = 5$$

$$4B = -51 + \frac{3}{4} = \frac{-207}{4}$$

$$B = -\frac{207}{16}$$

$$\therefore a_n = 4(4^n) - \frac{207}{16}\,n\,(4^n) + 8\,(5^n) + \frac{3}{16}\,n^2(4^n)$$

(5) (b) Solve the recurrence relation

$a_{n+2} + 4a_{n+1} + 4a_n = 7, \quad n \geq 0, \quad a_0 = 1, a_1 = 2.$

Solution

Let $a_n = cr^n$ be the solution of the homogeneous equation $a_{n+2} + 4a_{n+1} + 4a_n = 0$.

$$cr^{n+2} + 4cr^{n+1} + 4cr^n = 0$$

$$cr^n(r^2 + 4r + 4) = 0$$

$$r^2 + 4r + 4 = 0$$

$$(r+2)^2 = 0$$

$$r = -2, -2$$

$$a_n^{(h)} = \cancel{a_1 +} A(-2)^n + Bn(-2)^n$$

Since $f(n) = 7$, take $a_n^{(p)} = \cancel{Cn}\, C$

$$\cancel{C(n+2) + 4C(n+1) + 4Cn = 7}$$

$$C + 4C + 4C = 7$$

$$9C = 7$$

$$C = 7/9$$

$$\therefore \quad a_n = a_n^{(h)} + a_n^{(p)}.$$

$$a_n = A(-2)^n + Bn(-2)^n + 7/9.$$

(6) ~~(3)~~ Solve $a_{n+2} - 4a_{n+1} + 3a_n = -200, n \geq 0$, given

$a_0 = 3000, \quad a_1 = 3300.$

Solution

If $a_n^{(h)} = c\gamma^n$ be the solution of the homogeneous ~~equation~~ relation $a_{n+2} - 4a_{n+1} + 3a_n = 0$

$$a_{n+1}^{(h)} = c\gamma^{n+1}, \quad a_{n+2}^{(h)} = c\gamma^{n+2}.$$

$$c\gamma^{n+2} - 4c\gamma^{n+1} + 3c\gamma^n = 0$$

$$c\gamma^n(\gamma^2 - 4\gamma + 3) = 0$$

$$\gamma = 3, 1.$$

Hence $a_n^{(h)} = c_1(3^n) + c_2(1^n)$

$= c_1(3^n) + c_2$

Since $f(n) = -200 = -200(1^n)$ is a solution of the homogeneous ~~equation~~ relation, take

$a_n^{(p)} = An$ for some constant A.

$A(n+2) - 4A(n+1) + 3An = -200$

$-2A = -200$

$A = 100$

Hence $a_n = c_1(3^n) + c_2 + 100n$.

$a_0 = c_1 + c_2 = 3000$ —— (1)

$a_1 = 3c_1 + c_2 + 100 = 3300$

$3c_1 + c_2 = 3200$ —— (2)

(2) − (1) ⟹ $2c_1 = 200$

$c_1 = 100$

$c_2 = 3000 - 100 = 2900$

$a_n = 100(3^n) + 2900 + 100n$.

Exercises

Solve the recurrence relation

(1) $a_{n+1} - a_n = 2n+3$, $n \geq 0$, $a_0 = 1$

(2) $a_{n+1} - a_n = 3n^2 - n$, $n \geq 0$, $a_0 = 3$

(3) $a_{n+1} - 2a_n = 5$, $n \geq 0$, $a_0 = 1$

(4) $a_{n+1} - 2a_n = 2^n$, $n \geq 0$, $a_0 = 1$.

(5) $\cancel{a_{n+2} +}$ $a_{n+2} - 6a_{n+1} + 9a_n = 3(2^n) + 7(3^n)$